MEAN CONVERGENCE OF ORTHOGONAL FOURIER SERIES AND INTERPOLATING POLYNOMIALS

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Abstract. For a family of weight functions that include the general Jacobi weight functions as special cases, exact condition for the convergence of the Fourier orthogonal series in the weighted L^p space is given. The result is then used to establish a Marcinkiewicz-Zygmund type inequality and to study weighted mean convergence of various interpolating polynomials based on the zeros of the corresponding orthogonal polynomials.

1. Introduction

Let $d\alpha$ be a finite nonnegative measure on $[-1,1]$. We consider the Fourier orthogonal expansion with respect to $d\alpha$ and weighted L^p convergence of the interpolation polynomials based on the zeros of orthogonal polynomials with respect to $d\alpha$.

Throughout this paper we denote by $L^p(d\alpha)$ the space of measurable function f such that

$$\|f\|_{d\alpha,p} = \left(\int_{-1}^{1} |f(x)|^p d\alpha(x)\right)^{1/p}, \qquad 0 < p < \infty,$$

is finite. We assume that $\|f\|_\infty$ is the usual uniform norm for the continuous functions. If $d\alpha = wdt$, we may write $\|f\|_{w,p}$ instead of $\|f\|_{d\alpha,p}$. Let $p_n(d\alpha)$ denote the orthonormal polynomial of degree n with respect to the measure $d\alpha$ on $[-1,1]$. The zeros of $p_n(d\alpha)$ are distinct real numbers, denoted by $x_{1n}(d\alpha), x_{2n}(d\alpha), \ldots, x_{nn}(d\alpha)$, in $(-1,1)$. For any given function f on $[-1,1]$, we let $L_n(d\alpha; f)$ denote the unique Lagrange interpolation polynomial of degree $n-1$ that agrees with f at $x_{kn}(d\alpha)$, $1 \le k \le n$.

Let $d\beta$ be another measure defined on $[-1,1]$. We are interested in the precise condition on $d\beta$ and $d\alpha$ that will ensure the convergence of $L_n(d\alpha; f)$ in the $L^p(d\beta)$ norm for $f \in C[-1,1]$. This question was addressed by many authors (see [5, 6, 10, 11, 13] for historical account). In [6], Nevai solved the problem for the case that α' and β' are generalized Jacobi weight functions, defined by

$$(1.1) \qquad w(x) = h(x)\prod_{i=0}^{r+1} |x-t_i|^{\Gamma_i}, \qquad -1 = t_0 < t_1 < \ldots < t_r < t_{r+1} = 1,$$

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where h is a positive continuous function on $[-1,1]$ and the modulus of continuity ω of h satisfies $\int_0^1 (\omega(t)/t)dt < +\infty$. The condition for the convergence of $L_n(d\alpha; f)$ in $L^p(d\beta)$, $0 < p < \infty$, is given by

$$(\alpha' \sqrt{1-x^2})^{-p/2} \beta' \in L^1. \tag{1.2}$$

Since then this result has been extended in several directions, to other interpolation process and to more general weight functions (see, for example, [5, 6, 10, 11, 13] and the reference therein). It turned out ([13]) that one way of proving such results is to use a Marcinkiewicz-Zygmund type inequality. In the simplest case, such an inequality takes the form of

$$\|P\|_{d\beta,p} \le c \left(\sum_{k=1}^{n} c_{k,n} |P\left(x_{kn}(d\alpha)\right)|^p \right)^{1/p},$$

where P is any polynomial of degree at most $n-1$ and $c_{k,n}$ are certain (precisely known) nonnegative numbers. For α' and β' being generalized Jacobi weight functions, the inequality was proved in [13] under the precise condition (1.2) for the convergence of $L_n(d\alpha; f)$; furthermore, it was extended to include derivative values in the right hand side so that the convergence of Hermite interpolation polynomials can be derived.

We will try to establish the Marcinkiewicz-Zygmund type inequality for more general weight functions. To be sure, this has been done in [5]; but the result there requires an additional condition other than (1.2), namely, $(\alpha' \sqrt{1-x^2})^{q/2} \beta'^{q-1} \in L^1$, where $p^{-1} + q - 1 = 1$. To establish the inequality, one way is to use the boundedness of the orthogonal Fourier expansion. For $f \in L^2(d\alpha)$, such an expansion is given by

$$f \sim \sum_{n=0}^{\infty} c_n(f) p_n(d\alpha), \qquad c_n(f) = \int_{-1}^{1} f(t) p_n(d\alpha; t) d\alpha. \tag{1.3}$$

Let $S_n(d\alpha; f)$ denote the n-th partial sum of the expansion. The convergence of the Fourier expansion amounts to the uniform boundedness of $S_n(d\alpha; f)$. Finding the precise conditions for the boundedness of $\|S_n(d\alpha; f)\|_{d\beta,p}$ is an interesting problem in itself. By the Christoeff-Darboux formula, the kernel $K_n(x,y)$ of this operator has a singularity at $x = y$. To overcome the problem of the singularity, [13] established the following inequality

$$\int_{-1}^{1} \Big| \int_{-1}^{1} \frac{g(y)}{x-y}\, dy \Big|^p U^p(x)\, dx \le c \int_{-1}^{1} |g(x)|^p V^p(x)\, dx,$$

in which U and V are the generalized Jacobi weight functions in (1.1). This is an inequality for the Hilbert transform defined by

$$H(g; x) = \lim_{\varepsilon \to 0} \int_{|x-y| \ge \varepsilon} \frac{g(y)}{x-y} dy, \qquad g \in L^1.$$

In [3], double weight inequality for the Hilbert transform on $[0, \infty)$ is proved for general weight functions, the so-called A_p weight. Although the results in [3] do not apply to weight functions (1.1) directly, we show that they can be used to establish inequalities for weight functions that are more general than the weight function in (1.1) (see, for example, (2.2) and Definition 3.6), which in turn gives the result on mean convergence of various interpolating polynomials.

The paper is organized as follows: In the next section we fix the notation and state the preliminary. In section 3 we prove a double weight inequality for the Hilbert transform. The Fourier orthogonal series is studied in Section 4. The Marcinkiewicz-Zygmund inequalities are proved in Section 5, followed by the discussion on the mean convergence of interpolating polynomials in Section 6.

2. Notation and Preliminary

Throughout this paper we denote by Π_n the space of polynomials of degree at most n and by Π the space of all polynomials. We will use constants $c, c_1, c_2 \ldots$ to denote generic constants that depend only on weight functions and other fixed parameters involved, their values may vary from line to line. The notation $A \sim B$ means $|A^{-1}B| \le c$ and $|AB^{-1}| \le c$.

2.1. **Weight function.** First we define the weight functions that we shall deal with in this paper.

Definition 2.1. *A function w is called a generalized Jacobi weight function ($w \in GJ$ in short) if, for $t \in (-1,1)$,*

$$w(t) = h(t) \prod_{i=0}^{r+1} [\tau_i(|t-t_i|)]^{a_i}, \qquad -1 = t_0 < t_1 < \ldots < t_r < t_{r+1} = 1, \tag{2.1}$$

where $a_i \in \mathbb{R}$ and τ_i are nondecreasing, continuous semi-additive functions, $\tau_i(0) = 0$, h is a nonnegative function that satisfies $h \in L^\infty[-1,1]$ and $1/h \in L^\infty[-1,1]$ (we do not assume that $w \in L^1[-1,1]$).

A measure $d\alpha = \alpha' dt$ is called a GJ measure if $\alpha' \in GJ$ and $\alpha' \in L^1[-1,1]$.

Throughout the paper we often write $w \in GJ$ as

$$w(t) = h(t) \prod_{i=0}^{r+1} w_i(|t-t_i|), \qquad w_i(t) = [\tau_i(t)]^{a_i}, \tag{2.2}$$

and impose conditions on $w_i(t)$ instead on τ_i.

Definition 2.2. *Let $w \in GJ$ as in Definition 2.1. For $i = 0, 1, \ldots, r+1$, consider*

1. *$\tau_i(t)$ is concave; that is, $\tau_i(t) + \tau_i(s) \le 2\tau_i((t+s)/2)$;*
2. *$\int_0^\delta w_i(s) ds = \mathcal{O}(\delta w_i(\delta))$, $\quad \delta \to +0$;*
3. *$\omega(h;\delta)_\infty \delta^{-1} \in L^1[0,1]$ or $\omega(h,\delta)_2 = \mathcal{O}(\sqrt{\delta})$, $\quad \delta \to +0$, where ω is a modulus of continuity.*

We say $w \in GJ1$ if it satisfies (1), $w \in GJ2$ if it satisfies (2), and $w \in GJ3$ if it satisfies (3). We call w an admissible GJ weight function if it satisfies all three conditions.

If all $\tau_i(t) = t$, then $w \in GJ$ is precisely the usual generalized Jacobi weight function in (1.1), which is an admissible GJ weight function. Another family of GJ weight functions is given as follows.

Definition 2.3. *Let Γ_i and γ_i be real numbers. We denote by $GJ\log$ the collection of the GJ weight functions*

$$w(t) = h(t) \prod_{i=0}^{r+1} |t-t_i|^{\Gamma_i} \log^{\gamma_i} \frac{e}{|t-t_i|}. \tag{2.3}$$

The weight function $w \in GJ\log$ is in GJ2 if $\Gamma_i > -1$, $0 \le i \le r+1$ ([5, p. 328]). Furthermore, $\Gamma_i > -1$, $0 \le i \le r+1$ is necessary for $w \in GJ2$ and $w \in L^1$. We note that this condition is slightly stronger than $w \in L^1$. Also, $w \in GJ\log$ is in GJ1 if, say, $\gamma_i/\Gamma_i \ge 0$. We often write $\Gamma_i(w)$ or $\Gamma_i(d\alpha)$ in place of Γ_i to emphasis that they are parameters of w or $d\alpha$, respectively, of the above form.

For some results we also need one more restriction on the weight functions.

Definition 2.4. *If $w \in GJ$ can be written as $w = u/v$ such that both u and v are in the form of* (2.1) *with positive exponents, and v satisfies*

$$\int_0^\delta \frac{1}{v_i(s)} ds = \mathcal{O}\left(\frac{\delta}{v_i(\delta)}\right), \qquad 0 \le i \le r+1, \quad v(t) = h(t) \prod_{i=0}^{r+1} v_i(|t-t_i|),$$

then we say that $w \in GJ4$.

As an example, we point out that weight functions w in (2.3) are GJ4 weight functions if $\Gamma_i > -1$.

Throughout this paper, we reserve the notation φ for the functions

$$\varphi(x) = \sqrt{1-x^2} \quad \text{and} \quad \varphi(n,x) = \sqrt{1-x^2} + n^{-1}, \quad -1 \le x \le 1.$$

Furthermore, for $w \in GJ$ as in (2.2), we define

$$w(n,t) = \frac{w_0(\sqrt{1-t} + n^{-1}) w_{r+1}(\sqrt{1+t} + n^{-1})}{\sqrt{1-t^2} + n^{-1}} \prod_{i=1}^{r} w_i(|t-t_i| + n^{-1}). \tag{2.4}$$

2.2. **Orthogonal polynomials.** We consider orthonormal polynomials $p_n(d\alpha) \in \Pi_n$ with respect to $d\alpha = \alpha' dt$ and $\alpha' \in GJ$; that is,

$$\int_{-1}^1 p_n(d\alpha; t) p_m(d\alpha; t) d\alpha = \delta_{n,m}.$$

We assume that the zeros $x_{kn}(d\alpha)$ of $p_n(d\alpha)$ take the order

$$-1 < x_{nn}(d\alpha) < x_{n-1,n}(d\alpha) < \ldots < x_{1,n}(d\alpha) < 1.$$

The Christoeff function $\lambda_n(d\alpha)$ with respect to $d\alpha$ is defined by

$$\lambda_n(d\alpha; t) = \min_{P \in \Pi_{n-1}} \frac{1}{|P_n(t)|^2} \int_{-1}^1 |P(x)|^2 w(x) dx.$$

The numbers $\lambda_{kn}(d\alpha) = \lambda_n(d\alpha; x_{kn})$ are called the Cotes numbers, which appear in the Gauss quadrature formula

$$\sum_{k=1}^n P\big(x_{kn}(d\alpha)\big) \lambda_{kn}(d\alpha) = \int_{-1}^1 P(t) d\alpha, \qquad p \in \Pi_{2n-1}.$$

Further assumption on the weight function is needed to get bounds on these quantities.

Lemma 2.5. *Let $d\alpha$ be an admissible GJ measure. Then*

$$|p_n(d\alpha; x)| \le c\, \alpha'(n,x)^{-1/2} \varphi(n,x)^{-1/2}$$

uniformly for $-1 \le x \le 1$ and

$$|p_n'(d\alpha; x_{kn})| \sim \frac{n}{\varphi(n, x_{kn})} \frac{1}{\big[\alpha'(n, x_{kn}) \varphi(n, x_{kn})\big]^{1/2}}$$

uniformly for $1 \le k \le n$, *where* $x_{kn} = x_{kn}(d\alpha)$, *and*

$$\lambda_n(d\alpha; x) \sim \frac{1}{n}\alpha'(n,x)\varphi(n,x)$$

uniformly for $-1 \le x \le 1$.

For the proof of these estimates, see [1, 5]. It should be pointed out that some of the estimates hold for more general weight functions, or hold under weaker conditions for the weight functions in Definition 2.2. For example, for the estimate of $\lambda_n(d\alpha; x)$, only $\alpha' \in GJ2$ is needed. See the discussions in [5].

Let $w \in GJ$, for a fixed $d > 0$, we define $\Delta_n(\varepsilon)$ by

$$\Delta_n(\varepsilon) = [-1+\varepsilon n^{-2}, 1-\varepsilon n^{-2}] \Big\backslash \bigcup_{i=1}^{r} [t_i - \varepsilon n^{-1},\, t_i + \varepsilon n^{-1}]. \tag{2.5}$$

We shall use χ_E to denote the characteristic function of a set E. The following lemma is a simplified version of Theorem 3.5 in [5].

Lemma 2.6. *Let* $d\beta$ *be a GJ2 measure and* $u \in GJ$. *Then for each* $0 < p < +\infty$ *there exists an* $\varepsilon_0 > 0$ *such that for every fixed* ε, $0 < \varepsilon \le \varepsilon_0$, *and for* $P \in \Pi_n$,

$$\int_{-1}^{1} |P(t)|^p u(n,t) d\beta \le c \int_{\Delta_n(\varepsilon)} |P(t)|^p u(n,t) d\beta, \qquad n \ge n_0.$$

The next lemma gives an inequality for the quadrature sum of polynomials:

Lemma 2.7. *Let* α' *be an admissible GJ measure and* $v \in GJ2 \cap GJ4$. *Then for* $1 \le p < \infty$,

$$\sum_{k=1}^{n} \lambda_{kn}\left(v, x_{kn}(d\alpha)\right) |P(x_{k,n}(d\alpha))|^p \le c \int_{-1}^{1} |P(x)|^p v(x) dx$$

for every $P \in \Pi_{mn}$, *where* m *is a fixed positive integer and* c *is independent of* P *and* n.

Finally, there is the inequality of Bernstein-Markov type for general weight functions.

Lemma 2.8. *Let* $d\alpha$ *be a GJ4 measure and* $w \in GJ$. *Let* $0 < p < \infty$. *Then for arbitrary* $P \in \Pi_n$ *and integer* $j \ge 1$,

$$\int_{-1}^{1} \left| P^{(j)}(x)\varphi^j(n,x) \right|^p w(n,x)\varphi(n,x) d\alpha \le c n^{jp} \int_{-1}^{1} |P(x)|^p w(n,x)\varphi(n,x) d\alpha.$$

The last two results have been studied by several authors for various weight functions. In its present generality, it appears as [5, Theorem 3.6] and [5, Theorem 2.D]. See also [4] for some of the above inequalities with doubling weight.

3. Weighted inequalities for the Hilbert transform

We start with a result on the Hilbert transform proved in [3] for general weight functions defined on $[0,\infty)$.

Lemma 3.1. *Let* U *and* V *be nonnegative weight functions defined on* $[0,\infty)$, *and there exists a constant* A *such that either*

$$U(x) \le AU(y) \quad \text{and} \quad V(x) \le AV(y), \qquad x \le y \le 2x, \quad x > 0, \tag{3.1}$$

or the similar inequalities with $\ge$ in place of $\le$ hold. Let $1 < p < \infty$ and $fV \in L^p$. Then there is a constant c independent of f such that

$$\int_0^\infty \left| \int_0^\infty \frac{f(y)}{x-y} dy \right|^p U^p(x) dx \le c \int_0^\infty |f(x)|^p V^p(x) dx, \tag{3.2}$$

if for every interval $I \subset [0, \infty)$,

$$\left[\int_0^\infty \frac{|I|^{p-1} [U(x)]^p dx}{(|I| + |x - x_I|)^p} \right] \left[\frac{1}{|I|} \int_I [V(x)]^{-q} dx \right]^{p-1} \le B, \tag{3.3}$$

and

$$\left[\frac{1}{|I|} \int_I [U(x)]^p dx \right] \left[\int_0^\infty \frac{|I|^{q-1} [V(x)]^{-q}}{(|I| + |x - x_I|)^q} dx \right]^{p-1} \le B, \tag{3.4}$$

where $q = p/(p-1)$, $|I|$ denotes the length of I, x_I is the center of I and B is independent of I.

The lemma is stated in [3, Theorem 8] with the integral of the Hilbert transform in the left hand side. The proof there shows that the above version holds. By translation and truncation, it is possible to state a version of this theorem for the interval $[-1, 1]$. However, the condition (3.1) does not hold for GJ weight function. Our main result in this section is a double weight inequality for the GJ weight functions. The following two lemmas will be useful ([3, p. 281 and 282]).

Lemma 3.2. *Let $1 \le p < \infty$ and $fV \in L^1$. There is a finite c, independent of f, such that*

$$\int_0^\infty \left| \int_0^x f(t) dt \right|^p U^p(x) dx \le c \int_0^\infty |f(x)|^p V^p(x) dx$$

if and only if there is a finite B, independent of δ, such that for $\delta > 0$,

$$\left[\int_\delta^\infty [U(x)]^p dx \right] \left[\int_0^\delta [V(x)]^{-q} dx \right]^{p-1} \le B. \tag{3.5}$$

Lemma 3.3. *Let $1 \le p < \infty$ and $fV \in L^p$. There is a finite c, independent of f, such that*

$$\int_0^\infty \left| \int_x^\infty f(t) dt \right|^p U^p(x) dx \le c \int_0^\infty |f(x)|^p V^p(x) dx$$

if and only if there is a finite B, independent of δ, such that for $\delta > 0$,

$$\left[\int_0^\delta [U(x)]^p dx \right] \left[\int_\delta^\infty [V(x)]^{-q} dx \right]^{p-1} \le B. \tag{3.6}$$

Our main result in this section is the following theorem, in which U_i are parts of U as in the notation (2.2).

Theorem 3.4. *Let U and V be GJ weight functions. Let $1 < p < \infty$ and $gV \in L^p$. Then there is a constant c independent of f such that*

$$\int_{-1}^1 \left| \int_{-1}^1 \frac{g(y)}{x-y} dy \right|^p U^p(x) dx \le c \int_{-1}^1 |g(x)|^p V^p(x) dx, \tag{3.7}$$

if there is a B, independent of δ, such that for $\delta > 0$,

$$\left[\int_0^1 \frac{U_i^p(t) dt}{(\delta + t)^p} \right] \left[\int_0^\delta V_i^{-q}(t) dt \right]^{p-1} \le B, \qquad 0 \le i \le r+1, \tag{3.8}$$

and

$$\left[\int_0^\delta U_i^p(t)dt\right]\left[\int_0^1 \frac{V_i^{-q}(t)}{(t+\delta)^q}dt\right]^{p-1} \le B, \qquad 0 \le i \le r+1. \tag{3.9}$$

Proof. Recall that t_i are fixed numbers, $-1 = t_0 < t_1 < \ldots < t_r < t_{r+1} = 1$. We write

$$\begin{aligned}\int_{-1}^1 \Big| \int_{-1}^1 \frac{g(y)}{x-y}\,dy\Big|^p U^p(x)\,dx &= \sum_{i=0}^r \int_{t_i}^{t_{i+1}} \Big| \int_{-1}^1 \frac{g(y)}{x-y}\,dy\Big|^p U^p(x)\,dx \\ &\le c\sum_{i=0}^r \int_{t_i}^{t_{i+1}} \Big| \int_{-1}^1 \frac{g(y)}{x-y}\,dy\Big|^p U_i^p(|x-t_i|)U_{i+1}^p(|x-t_{i+1}|)dx.\end{aligned}$$

For each i, we then break the inner integral of the last expression into three integrals over $(-1,t_i)$, (t_i,t_{i+1}) and $(t_{i+1},1)$, respectively, and estimate the corresponding terms separately.

We estimate the middle part first. Changing variables

$$x - t_i = \frac{t_{i+1}-t_i}{1+X} \quad , \quad y - t_i = \frac{t_{i+1}-t_i}{1+Y}$$

in the integrals gives

$$\begin{aligned}J_i &:= \int_{t_i}^{t_{i+1}} \Big| \int_{t_i}^{t_{i+1}} \frac{g(y)}{x-y}\,dy\Big|^p U_i^p(|x-t_i|)U_{i+1}^p(|x-t_{i+1}|)dx \\ &= \int_0^{+\infty} \Big| \int_0^{+\infty} \frac{g(y)}{X-Y}\frac{X+1}{Y+1}dY\Big|^p U_i^p\left(\frac{t_{i+1}-t_i}{X+1}\right) U_{i+1}^p\left(\frac{t_{i+1}-t_i}{X+1}X\right)\frac{t_{i+1}-t_i}{(X+1)^2}dX.\end{aligned}$$

If we can apply Lemma 3.1 with $f(Y) = g(y)/(1+Y)$ and with u and v in place of U and V, where

$$\begin{aligned}u^p(X) &:= U_i^p\left(\frac{t_{i+1}-t_i}{X+1}\right) U_{i+1}^p\left(\frac{t_{i+1}-t_i}{X+1}X\right)(X+1)^{p-2} \\ v^p(X) &:= V_i^p\left(\frac{t_{i+1}-t_i}{X+1}\right) V_{i+1}^p\left(\frac{t_{i+1}-t_i}{X+1}X\right)(X+1)^{p-2},\end{aligned}$$

then we will end up with the desired estimate

$$\begin{aligned}J_i &\le c\int_0^{+\infty} |g(x)|^p\Big|^p V_i^p\left(\frac{t_{i+1}-t_i}{X+1}\right) V_{i+1}^p\left(\frac{t_{i+1}-t_i}{X+1}X\right)\frac{t_{i+1}-t_i}{(X+1)^2}dX \\ &= c\int_{t_i}^{t_{i+1}} |g(y)|^p V_i^p(|y-t_i|)V_{i+1}^p(|y-t_{i+1}|)dy \le c\int_{t_i}^{t_{i+1}} |g(y)|^p V^p(y)dy.\end{aligned}$$

The functions U_i and V_i satisfy (3.1) automatically, since τ_i in the Definition 2.1 is nondecreasing so that U_i and V_i are either nondecreasing or nonincreasing. Thus, we only need to verify that the conditions (3.3) and (3.4) of Lemma 3.1 are satisfied with u and v in place of U and V. First we note that the change of variables leads to $u^p(x)/(1+X)^p \in L^1[0,\infty)$ and $v^{-q}(x)/(1+X)^q \in L^1[0,\infty)$, since (3.8) and (3.9) with a fixed δ shows that $U^p \in L^1$ and $V^{-q} \in L^1$. Hence, if $|I|$ is fixed, then the inequality $|I| + |X - X_I| \ge c(1+X)$ shows that the (3.3) and (3.4) holds trivially. Thus, we need to consider only the following two cases:

Case 1. $I=(0,2\delta)$, $\delta<1/2$. Using the fact that $|I|+|X-X_I|=2\delta+|X-\delta|\ge(1+X)/2$ if $X>1$, the left hand side of (3.3) is bounded by

$$c+\left(\int_0^1\frac{[u(X)]^p}{(\delta+|X-\delta|)^p}dX\right)\left(\int_0^{2\delta}[v(X)]^{-q}dX\right)^{p-1}$$
$$\le c+c\left(\int_0^1\frac{\left[U_{i+1}\left(\frac{t_{i+1}-t_i}{1+X}X\right)\right]^p}{(2\delta+|X-\delta|)^p}dX\right)\left(\int_0^{2\delta}\left[V_{i+1}\left(\frac{t_{i+1}-t_i}{1+X}X\right)\right]^{-q}dX\right)^{p-1}$$
$$\le c+c\left(\int_0^1\frac{[U_{i+1}(x)]^p}{(x+\delta)^p}dx\right)\left(\int_0^{2\delta}[V_{i+1}(x)]^{-q}\,dx\right)^{p-1}.$$

The last expression is bounded by (3.9). The inequality (3.4) in the case of $I\subset(0,2\delta)$ is established similarly.

Case 2. $I=(s,R)$, $s<R$, R can be arbitrarily large. If $X\le R/2$, then $|I|+|X-X_I|=(R-s)+|X-(R+s)/2|\ge R-s/2\ge X+1$; so that

$$\int_0^{R/2}\frac{[u(X)]^p}{(\delta+|X-\delta|)^p}dX\le\int_0^{R/2}\frac{[u(X)]^p}{(1+X)^p}dX\le c,$$

as $u^p/(1+X)^p\in L^1[0,\infty)$. The left had side of (3.3) is bounded since changing variables from X and Y back to x and y shows that

$$\int_{R/2}^\infty\frac{[u(X)]^p}{(\delta+|X-\delta|)^p}dX\le c\int_{R/2}^\infty\frac{[U_i\left(\frac{t_{i+1}-t_i}{1+X}\right)]^p}{(\delta+|X-\delta|)^p}(1+X)^{p-2}dX$$
$$\le c\delta^p\int_{t_i}^{t_i+\delta}\frac{[U_i(|x-t_i|)]^p}{(\delta+|x-t_i|)^p}dx=c\int_0^\delta[U_i(t)]^pdt,$$

where $\delta=(t_{i+1}-t_i)/(1+R/2)$ and we have used the fact that

$$(R-s)+|X-(R+s)/2|\ge R/2+X-s\ge(R+X+1)/4,$$

and since by $-q(p-2)/p=q-2$,

$$\int_s^R[v(X)]^{-q}dX\le\int_s^R\left[V_i\left(\frac{t_{i+1}-t_i}{1+X}\right)\right]^{-q}(1+X)^{q-2}dX$$
$$\le c\int_{t_i+\delta}^{t_i+\rho}[V_i(|x-t_i|)]^{-q}\frac{dx}{|x-t_i|^q}\le c\int_\delta^\rho[V_i(t)]^{-q}\frac{dt}{t^q}\le c\int_0^1\frac{[V_i(t)]^{-q}}{(t+\delta)^q}dt,$$

where $\rho=(t_{i+1}-t_i)/(1+R)$ and δ is as above, so that the boundedness of the left hand side of (3.3) follows from (3.9). The inequality (3.4) is established similarly. Consequently, we have justified the use of Lemma 3.1 and the bound of J_i.

Next, let $\bar t_i=(t_i+t_{i+1})/2$. We split the first integral as follows,

$$\int_{t_i}^{t_{i+1}}\left|\int_{-1}^{t_i}\frac{g(y)}{x-y}dy\right|^pU_i(|x-t_i|)U_{i+1}^p(|x-t_{i+1}|)dx$$
$$\le c\int_{\bar t_i}^{t_{i+1}}\left[\int_{-1}^{t_i}|g(y)|dy\right]^pU_{i+1}^p(|x-t_{i+1}|)dx$$
$$+c\int_{t_i}^{\bar t_i}\left[\int_{-1}^{\bar t_{i-1}}|g(y)|dy\right]^pU_i^p(|x-t_i|)dx+c\int_{t_i}^{\bar t_i}\left|\int_{\bar t_{i-1}}^{t_i}\frac{g(y)}{x-y}dy\right|^pU_i^p(|x-t_i|)dx.$$

The first two terms can be estimated by the Hölder inequality. For example, the first term is bounded by

$$c\int_{-1}^{t_i}|g(y)|^pV^p(y)dy\left[\int_{-1}^{t_i}[V(y)]^{-q}\,dy\right]^{p/q}\int_{\bar t_i}^{t_{i+1}}U_{i+1}^p(|x-t_{i+1}|)dx$$
$$\le c\int_{-1}^{t_i}|g(y)|^pV^p(y)dy.$$

The third term needs more work. Changing variables $x-t_i=X$, $y-t_i=-Y$, it becomes a constant multiple of

$$L:=\int_0^A\Big|\int_0^B\frac{g(y)}{X+Y}dY\Big|^pU_i^p(X)dX,\qquad A=(t_{i+1}-t_i)/2,\quad B=(t_i-t_{i-1})/2.$$

To estimate this term we need to use Lemma 3.2 and Lemma 3.3. Changing variables

$$X=\frac{As}{1+s}\qquad\text{and}\qquad Y=\frac{At}{1+t},$$

and splitting the inner integral into two parts gives

$$L=AB^p\int_0^\infty\Big|\int_0^\infty\frac{g(y)}{As(1+t)+Bt(1+s)}\frac{1+s}{1+t}dt\Big|^pU_i^p\left(\frac{As}{1+s}\right)\frac{ds}{(1+s)^2}$$
$$\le A^{1-p}B^p\int_0^\infty\left[\int_0^s\frac{|g(y)|}{s(1+t)}\frac{1+s}{1+t}dt\right]^pU_i^p\left(\frac{As}{1+s}\right)\frac{ds}{(1+s)^2}$$
$$+A\int_0^\infty\left[\int_s^\infty\frac{|g(y)|}{t(1+s)}\frac{1+s}{1+t}dt\right]^pU_i^p\left(\frac{As}{1+s}\right)\frac{ds}{(1+s)^2}.$$

To estimate the first integral in the right hand side, we use Lemma 3.2 with $f(t)=g(y)/(1+t)^2$ and with u and v in place of U and V, where

$$u^p(s)=U_i^p\left(\frac{As}{1+s}\right)\left(\frac{1+s}{s}\right)^p\frac{1}{(1+s)^2}\quad\text{and}\quad v^p(s)=V_i^p\left(\frac{Bt}{1+t}\right)(1+t)^{2(p-1)},$$

so that the term is bounded by, after changing the integral back to y,

$$c\int_0^\infty\left|\frac{g(y)}{(1+t)^2}\right|^pV_i^p\left(\frac{Bt}{1+t}\right)(1+t)^{2p}\frac{dt}{(1+t)^2}$$
$$=c\int_0^B|g(y)|^pV_i^p(Y)dY=c\int_{\bar t_{i-1}}^{t_i}|g(y)|^pV_i^p(|y-t_i|)dy.$$

The condition of Lemma 3.2 is verified as follows: changing variables back to X and Y,

$$\int_\delta^\infty u^p(s)ds\left(\int_0^\delta v^{-q}(t)dt\right)^{p-1}=\left(\frac{A}{B}\right)^{p-1}\int_{A\delta}^A\frac{U_i^p(X)}{X^p}dX\left(\int_0^{B\delta}V_i^{-q}(Y)dY\right)^{p-1}$$
$$\le c\left(\frac{A}{B}\right)^{p-1}\int_0^A\frac{U_i^p(x)}{(x+\delta)^p}dx\left(\int_0^{B\delta}V_i^{-q}(y)dy\right)^{p-1},$$

which is bounded by a constant by (3.8). The second term is estimated using Lemma 3.3 with $f(t)=g(y)/(t(1+t))$ and with u and v in place of U and V, where

$$u^p(s)=U_i^p\left(\frac{As}{1+s}\right)\frac{1}{(1+s)^2}\quad\text{and}\quad v^p(s)=V_i^p\left(\frac{Bt}{1+t}\right)t^p(1+t)^{p-2},$$

so that the term is bounded by, after changing the integral back to y,

$$c\int_0^\infty \left|\frac{g(y)}{t(1+t)}\right|^p V_i^p\left(\frac{Bt}{1+t}\right)(t(1+t))^p\frac{dt}{(1+t)^2}$$
$$= c\int_0^B |g(y)|^p V_i^p(Y)dY = c\int_{\bar t_{i-1}}^{t_i} |g(y)|^p V_i^p(|y-t_i|)dy.$$

The condition of Lemma 3.3 is verified similarly; it reduces to the condition (3.9). Putting these estimates together gives the stated inequality (3.7). □

For U and V being the classical GJ weight functions, the inequality (3.7) was proved in [13] under the conditions

$$U^p \in L^1, \qquad V^{-q} \in L^1, \qquad U(x) \le cV(x).$$

A more general result along this line is the following:

Proposition 3.5. *Let $1 < p < \infty$. Let U and V be GJ $\log$ weight functions in Definition 2.3. Then the inequality* (3.7) *holds if $p\Gamma_i(U) > -1$ or $-q\Gamma_i(V) > -1$ and $U(t) \le cV(t)$.*

Proof. The condition $U(x) \le cV(x)$ is equivalent to $\Gamma_i(U) \ge \Gamma_i(V)$, and $\gamma_i(U) \le \gamma_i(V)$ when $\Gamma_i(U) = \Gamma_i(V)$. That $U^p \in L^1$ implies either $p\Gamma_i(U) > -1$ or $p\Gamma_i(U) = -1$ and $p\gamma_i(U) > -1$. Hence, $p\Gamma_i(U) > -1$ or $-q\Gamma_i(V) > -1$ implies that $U^p \in L^1$ or $V^{-q} \in L^1$, respectively. We show that the U and V satisfy (3.8) and (3.9) under the given conditions.

As it is shown in [5, p. 328], that $p\Gamma_i(U) > -1$ and $-q\Gamma_i(V) > -1$ shows U_i and V_i satisfy

$$\int_0^\delta U_i^p(t)dt = \mathcal{O}\left(\delta U_i^p(\delta)\right) \quad \text{and} \quad \int_0^\delta V_i^{-q}(t)dt = \mathcal{O}\left(\delta V_i^{-q}(\delta)\right). \tag{3.10}$$

Hence, $U_i(x) \le cV_i(x)$ shows that

$$\int_0^\delta \frac{U_i^p(t)}{(t+\delta)^p}dt\left(\int_0^\delta V_i^{-q}(t)dt\right)^{p-1} \le \delta^{-p}\int_0^\delta U_i^p(U)dt\,\left(\delta V_i^{-q}(\delta)\right)^{p-1}$$
$$\le c\,U_i^p(\delta)V_i^{-p}(\delta) \le c.$$

Note that the part of $(\int_0^\delta V_i^{-q}(t)dt)^{p-1}$ in (3.8) is finite since $V^{-q} \in L^1$, and it is always bounded by $V_i^{-p}(\delta)\delta^{p-1} = \mathcal{O}(\delta^\varepsilon)$ for some $\varepsilon > 0$. We estimate the integral of U_i^p on $(\delta, 1)$,

$$\int_\delta^1 \frac{U_i^p(t)}{(\delta+t)^p}dt \le \int_\delta^1 \frac{U_i^p(t)}{t^p}dt.$$

If $(\Gamma_i(U)-1)p > -1$, then $U^p/t^p \in L^1$ so that (3.8) holds trivially. If $(\Gamma_i(U)-1)p = -1$, then

$$\int_\delta^1 \frac{U_i^p(t)}{t^p}dt = \int_\delta^1 \left(\log\frac{e}{t}\right)^{\gamma_i(U)p}\frac{dt}{t} = \frac{(\log(e/\delta))^{\gamma_i(U)p+1}-1}{\gamma_i(U)p+1}.$$

In this case the fact that $V_i^{-p}(\delta)\delta^{p-1} = \mathcal{O}(\delta^\varepsilon)$ shows that (3.8) holds. Finally, if $(\Gamma_i(U)-1)p < -1$, we show that $\int_\delta^1 (U_i^p(t)/t^p)dt = \mathcal{O}\left(\delta^{-p+1}U_i^p(\delta)\right)$ to finish

the proof (see also [5, p. 328]). In this case, if $\gamma_i(U)p \ge 0$, then since $\log(e/t)$ is decreasing,

$$\int_\delta^1 \frac{U_i^p(t)}{t^p} dt = \int_\delta^1 t^{p\Gamma_i(U)-p} \left(\log \frac{e}{t}\right)^{p\gamma_i(U)} dt$$
$$\le c \left(\log \frac{e}{\delta}\right)^{p\gamma_i(U)} \int_\delta^1 t^{p\Gamma_i(U)-p} dt \le c\, \delta^{-p+1} U_i^p(\delta),$$

since $p\Gamma_i(U) - p + 1 < 0$. If $\gamma_i(U)p < 0$, we choose $\varepsilon > 0$ such that $(\Gamma_i(U) - 1)p - \varepsilon\gamma_i(U)p < -1$. Then since $t^\varepsilon \log(e/t)$ is increasing for t close to zero, we have

$$\int_\delta^1 \frac{U_i^p(t)}{t^p} dt = \int_\delta^1 t^{p\Gamma_i(U)-p-\varepsilon\gamma_i(U)p} \left(t^\varepsilon \log \frac{e}{t}\right)^{p\gamma_i(U)} dt$$
$$\le c \left(\delta^\varepsilon \log \frac{e}{\delta}\right)^{p\gamma_i(U)} \int_\delta^1 t^{p\Gamma_i(U)-p-\varepsilon\gamma_i(U)p} dt \le c\, \delta^{-p+1} U_i^p(\delta).$$

Putting these estimates together, we have verified (3.8). The inequality (3.9) can be verified similarly. □

Remark 3.1. It should be pointed out that the condition $p\Gamma_i(U) > -1$ and $-q\Gamma_i(V) > -1$ cannot be replaced by $U^p \in L^1$ and $V^{-q} \in L^1$ without further restriction on $\gamma_i(U)$ and $\gamma_i(V)$. Assume, for example, that $-q\Gamma_i(V) = -1$. Then $V^{-q} \in L^1$ holds if $-q\gamma_i(V) < -1$, and we have

$$\int_0^\delta V_i^{-q}(t) dt = \int_0^\delta \left(\log \frac{e}{t}\right)^{-q\gamma_i(V)} \frac{dt}{t} = \frac{(\log(e/\delta))^{-q\gamma_i(V)+1}}{q\gamma_i(V) - 1}.$$

Here $\Gamma_i(U) \ge \Gamma_i(V) = 1/q$. If $\Gamma_i(U) = 1/q$ then $(\Gamma_i(U) - 1)p = -1$ and the left hand side of (3.8) is bounded by

$$\frac{(\log(e/\delta))^{\gamma_i(U)p+1} - 1}{\gamma_i(U)p + 1} \left(\frac{(\log(e/\delta))^{-q\gamma_i(V)+1}}{q\gamma_i(V) - 1}\right)^{p-1} = c \left(\log \frac{e}{\delta}\right)^{p(\gamma_i(U)-\gamma_i(V)+1)},$$

which is bounded only if $\gamma_i(U) - \gamma_i(V) + 1 \le 0$. □

Using the same argument, by induction if necessary, one can establish the similar result for weight functions of the type $t^{\Gamma_i}(\log_k(e/t))^{\gamma_i}$, where $\log_k t = \log\log\ldots\log t$ (k fold of log). To state a more general result, we need the following definition:

Definition 3.6. *A function S is called slowly varying if for any $\varepsilon > 0$, $t^\varepsilon S(t) \to \infty$ and $t^{-\varepsilon} S(t) \to 0$ as $t \to \infty$.*

For example, $S(t) = \log(e/t)$ is a slowly varying function and so is a power of $S(t)$. Also, $(\log\log e/t)^\gamma$, and more generally, the powers of $\log_k e/t$ are all slowly varying functions. A slightmodification of the proof of Proposition 3.5 gives the following result:

Proposition 3.7. *Let S_{U_i} and S_{V_i} be slowly varying functions such that either they are increasing functions or, for any $\varepsilon > 0$, $t^{-\varepsilon} S_{U_i}(1/t)$ and $t^{-\varepsilon} S_{V_i}(1/t)$ are increasing for small $t > 0$. Then the inequality (3.7) holds if U and V are GJ2 weight with*

$$U_i(t) = t^{\Gamma_i(U)} S_{U_i}(1/t) \quad \textit{and} \quad V_i(t) = t^{\Gamma_i(U)} S_{V_i}(1/t), \qquad 0 \le i \le r+1,$$

such that $p\Gamma_i(U) > -1$, $-q\Gamma_i(V) > -1$, and $U(x) \le cV(x)$.

The assumption that U and V are in GJ2 replaces the condition (3.10). The Proposition 3.5 corresponds to the case that both S_{U_i} and S_{V_i} are of the form $(\log(et))^{p\gamma_i}$. If $\gamma_i \le 0$, this function is increasing; if $\gamma_i \ge 0$, then $\varepsilon^{-\varepsilon}(\log(e/t))^{p\gamma_i}$ is increasing.

Let us mention that the definition of the slowly varying functions may be different in the literature. For example, in [9], it is defined as the functions that satisfy the relation

$$\lim_{t\to\infty} \frac{S(\lambda t)}{S(t)} = 1, \qquad \text{for each} \quad \lambda > 0.$$

Clearly, our condition is more relaxed. A slowly varying function S that satisfy the above limiting condition also satisfies the following property ([9]): If S is defined on $[a,\infty)$, $a > 0$, then there exists a $b \ge a$ such that for all $t \ge b$

$$S(t) = \exp\left[\eta(t) + \int_b^t \frac{\epsilon(x)}{x} dx\right],$$

where η is a bounded measurable function on $[b,\infty)$ such that $\eta(t) \to c$ $(|c| \to \infty)$, and ϵ is a continuous function on $[b,\infty)$ such that $\epsilon(t) \to 0$ as $t \to \infty$.

4. Convergence of orthogonal series

Let $d\alpha$ be a GJ measure and we assume that α' is in the form of (2.2)

$$\alpha'(x) = h(t)\prod_{i=0}^{r+1} \alpha_i(|t-t_i|), \qquad \alpha_i(t) = [\tau_i(t)]^{a_i}.$$

Let $S_n(d\alpha; f)$ be the partial sum of the Fourier orthogonal series. By (1.3),

$$S_n(d\alpha; f, x) = \sum_{k=0}^{n-1} c_k(f) p_k(d\alpha) = \int_{-1}^{1} f(y) K_n(d\alpha; x, y) d\alpha(y).$$

The kernel $K_n(d\alpha; x, t)$, by the Christoffel-Darboux formula, satisfies the formula

$$K_n(d\alpha; x, y) = a_{n-1} \frac{p_n(d\alpha; x)p_{n-1}(d\alpha; y) - p_n(d\alpha; y)p_{n-1}(d\alpha; x)}{x-y},$$

where a_{n-1} is a proper constant. Our main results on the mean convergence of the generalized Jacobi series are the following (cf. [13, I, p. 246]).

Theorem 4.1. *Let $d\alpha$ be an admissible GJ measure and assume that α_i are nondecreasing for $1 \le i \le r$ and that $\alpha_i\varphi^3$ are nondecreasing for $i = 0$ and $r+1$. Let $u, w \in GJ$. Define U and V by*

$$U^p := w^p(\alpha'\varphi)^{-p/2}\alpha' \quad \text{and} \quad V^{-q} := \varphi^q u^{-q}(\alpha'\varphi)^{-q/2}\alpha'.$$

Assume that U^p and V^{-q} satisfy (3.8) and (3.9). Let $1 < p < +\infty$. Then

$$\|S_n(d\alpha, f)w\|_{d\alpha,p} \le c\|fu\|_{d\alpha,p} \tag{4.1}$$

for every f such that $\|fu\|_{d\alpha,p} < +\infty$ if and only if

$$\begin{aligned} &w^p\alpha' \in L^1 \quad , \quad u^{-q}\alpha' \in L^1 \ , \\ &w^p\big(\alpha'\varphi\big)^{-p/2}\alpha' \in L^1 \quad , \quad u^{-q}\big(\alpha'\varphi\big)^{-q/2}\alpha' \in L^1 \end{aligned} \tag{4.2}$$

and

$$w(x) \le cu(x). \tag{4.3}$$

Proof. Let $q_n(x)$ denote the orthonormal polynomials associated with the measure $(1-x^2)\,d\alpha(x)$; that is, $q_n(x) = p_n(\varphi^2)\,d\alpha, x)$. Let

$$h_1(x,y) = p_n(d\alpha,x)p_n(d\alpha,y),$$
$$h_2(x,y) = \frac{F_n(x,y)}{x-y}, \qquad F_n(x,y) = (1-y^2)p_n(d\alpha,x)q_{n-1}(y)$$
$$h_3(x,y) = h_2(y,x).$$

Following Pollard [7], the kernel $K_n(d\alpha;x,t)$ can be written as

$$K_n(d\alpha;x,y) = \alpha_n h_1(x,y) + \beta_n h_2(x,y) + \beta_n h_3(x,y),$$

where the numbers α_n and β_n depend on $d\alpha$ and n. Since $\alpha' > 0$, a.e., it follows from [8] that $|\alpha_n|$ and $|\beta_n|$ are bounded by constant independent of n (cf. [7, p. 358-360]). Thus, it is sufficient to prove that

$$\int_{-1}^{1}\left|\int_{-1}^{1} h_k(x,y)f(y)\,d\alpha(y)\right|^p w^p(x)\,d\alpha(x) \le c\|fu\|^p_{d\alpha,p} \tag{4.4}$$

for $k = 1, 2$ and 3 under the conditions (4.2) and (4.3).

The bound of $p_n(d\alpha)$ in Lemma 2.5 shows, in particular, that

$$|p_n(d\alpha;x)| \le c\left[1 + (\alpha'(x)\varphi(x))^{-1/2}\right].$$

A similar estimate also applies to $q_n(y)$. Applying the Hölder inequality to the inner integral and then using the bounds of orthogonal polynomials, it follows readily that the inequality (4.4) holds for $k = 1$ under the condition (4.2).

To prove (4.4) for $k = 2$, we use Theorem 3.4. First, by Lemma 2.6 (recall $\alpha' \in GJ2$) and the fact that $S_n(d\alpha;f,x)$ is a polynomial of degree n, it is sufficient to prove that

$$\int_{-1}^{1}\left|\int_{-1}^{1} f(y)\frac{F_n(x,y)}{x-y}d\alpha(y)\right|^p w^p(x)\chi_{\Delta_n(\varepsilon)}(x)\,d\alpha(x) \le c\|fu\|^p_{d\alpha,p} \tag{4.5}$$

under the conditions (4.2) and (4.3). Lemma 2.5 shows that

$$|p_n(x)| \le c\big(\alpha'(x)\varphi(x)\big)^{-1/2}, \qquad x \in \Delta_n(\varepsilon).$$

Furthermore, the assumption that α_i is nondecreasing for $0 \le i \le r$ and $\alpha_i\varphi^3$ is nondecreasing for $i = 0$ and $i = r+1$ shows that q_n is bounded by

$$|q_n(y)| \le c\big(\alpha'(n,y)\varphi^3(n,y)\big)^{-1/2} \le c\big(\alpha'(y)\varphi^3(y)\big)^{-1/2}, \qquad y \in (-1,1). \tag{4.6}$$

Hence, by the definition of $F_n(x,y)$, the left hand side of (4.5) is bounded by

$$c\int_{-1}^{1}\left|\int_{-1}^{1}\frac{f(y)(\alpha'(y)\varphi^3(y))^{-1/2}\phi_n(y)}{x-y}\varphi^2(y)d\alpha(y)\right|^p (\alpha'(x)\varphi(x))^{-p/2}w^p(x)d\alpha(x),$$

where $\phi_n(y)$ is a function bounded by a constant independent of n, which is bounded by $c\|fu\|^p_{d\alpha,p}$ upon using Theorem 3.4 with $f(\alpha'\varphi^3)^{-1/2}\phi_n$ in place of g.

For $k = 3$, we use a dual argument and derive the desired bound from the case $k = 2$. This argument does not depend on the fact that our weight functions are the generalized Jacobi ones. We refer to [12, p. 889] for the details. Thus, the proof for the sufficient part is completed.

The conditions (4.2) are proved to be necessary for general weight functions in [2]. The condition (4.3) is necessary also for general weight functions as the proof in [13, p. 250] shows. □

Since $S_n(d\alpha, f)$ is a projection operator, Theorem 4.1 and Weierstrauss theorem give the following corollary:

Corollary 4.2. *Under the assumptions of Theorem 4.1*

$$\lim_{n\to\infty} \|\big(S_n(d\alpha, f) - f\big)w\|_{d\alpha,p} = 0$$

for every f such that $\|fu\|_{d\alpha,p} < \infty$ if and only if (4.2) *and* (4.3) *hold.*

Remark 4.1. The assumption that U^p and V^{-q} satisfy (3.8) and (3.9) already implies that $U^p \in L^1$ and $V^{-q} \in L^1$. Hence, the condition $w^p(\alpha'\varphi)^{-p/2}\alpha' \in L^1$ in (4.2) is redundant for the sufficient part. We include it since it is also a necessary condition. Note also that $u^p(\alpha'\varphi)^{-p/2}\alpha' = \varphi^{-q}V^{-q}$ so that the condition $u^p(\alpha'\varphi)^{-p/2}\alpha' \in L^1$ in (4.2) is still needed.

Remark 4.2. Since $\alpha_i(x) = [\tau_i(x)]^{a_i}$ and τ_i are nondecreasing by definition, the assumption on the nondecreasing of α_i holds if a_i are nonnegative. This assumption is not needed for the ordinary GJ weight functions in (1.1). See the discussion after the proof of Corollary 4.3.

Corollary 4.3. *Let $d\alpha$ be an admissible GJ measure. Assume that α', u and v are in $GJ\log$ such that, for $1 \le i \le r$, $\Gamma_i(\alpha') > 0$ or $\Gamma_i(\alpha') = 0$ and $\gamma_i(\alpha) \le 0$. Let U and V be defined as in Theorem 4.1 and assume that $p\Gamma_i(U) > -1$ and $-q\Gamma_i(V) > -1$ for $0 \le i \le r+1$. Then the inequality* (4.1) *holds if and only if* (4.2) *and* (4.3) *hold.*

Proof. The assumption on α' means that $\alpha_i'(t) = t^{\Gamma_i(\alpha')}\,(\log(e/t))^{\gamma_i(\alpha')}$. It is easy to see that this function is nondecreasing if $\Gamma_i(\alpha') > 0$ or if $\Gamma_i(\alpha') = 0$ and $\gamma_i(\alpha') \le 0$. Since $\alpha' \in L^1$ implies that $\Gamma_i(\alpha') > -1$, this shows that $\alpha_i'\varphi^3$ is increasing for $i = 0$ and $r+1$ without further conditions. The assumption that $p\Gamma_i(U) > -1$ and $-q\Gamma_i(V) > -1$ implies, by Proposition 3.5, that U^p and V^{-q} satisfy (3.8) and (3.9). □

Again, we note that the $p\Gamma_i(U) > -1$ implies that $U^p \in L^1$ so that part of the (4.2) is redundant. If all $\gamma_i = 0$ in the above corollary, then it deals with the ordinary GJ weight function in (1.1). In that case, the assumption $p\Gamma_i(U) > -1$ and $-q\Gamma_i(V) > -1$ are not needed, and the result was proved in [1] for the case of $w = u$ and in [13] for the case of $w \ne u$. However, the results there were proved without the assumption that $\Gamma_i \ge 0$ for $1 \le i \le r$ (recall that $\alpha' \in L^1$ implies $\Gamma_i > -1$). The additional assumption is used to ensure that (4.6) holds for all $y \in (-1,1)$. For weight functions in the Corollary 4.3, it is possible to follow the proof in [13] to remove the additional assumption. This amounts to apply the current proof to $f(y)\chi_{\tau_n(\varepsilon)}$, where $\tau_n(\varepsilon)$ is the set

$$\tau_n(\varepsilon) := [-1,1] \setminus \bigcup_{i\in\sigma} [t_i - \varepsilon n^{-1}, t_i + \varepsilon n^{-1}],$$

in which $\sigma = \{i : \Gamma_i(\alpha') < 0 \quad \text{or} \quad \Gamma_i(\alpha') = 0, \gamma_i(\alpha') > 0\}$ (since the inequality (4.6) holds for $y \in \tau_n(\varepsilon)$), and show that

$$\int_{-1}^1 \Big| \int_{-1}^1 h_2(x,y) f(y)\big(1 - \chi_{\tau_n(d)}(y)\big)\, d\alpha(y)\Big|^p w^p(x)\chi_{\Delta_n(\varepsilon)}(x)\, d\alpha(x) \le c\|fu\|^p_{d\alpha,p}$$

separately. The second part involves difficult estimates that have to be worked out. It is not clear how to extend this part to general GJ weight functions in Theorem 4.1.

Remark 4.3. Under the additional assumption that $f\alpha_i$ is continuous locally at $x = t_i$, $1 \le i \le r$, we can remove the condition that α_i are nondecreasing for $1 \le i \le r$ from the assumption of the Theorem 4.1. We then have $\|S_n(d\alpha, f)w\|_{d\alpha,p}$ is uniformly bounded for every f such that $\|fu\|_{d\alpha,p} < +\infty$ and f locally continuous at t_i under the conditions (4.2) and (4.3). For the proof, we can assume that α_i is decreasing, since the other case has been settled in the proof of Theorem 4.1. Then it follows from the estimate in Lemma 2.5 that $|(1-y^2)q_{n-1}(y)| \le c$. Hence, we can use the fact that

$$\lim_{n\to\infty} n \int_{t_i - n^{-1}}^{t_i + n^{-1}} |f(t)\alpha'(t)| dt = |f(t_i)\alpha'(t_i)|,$$

which holds since $f\alpha'$ is locally continuous at t_i to deal with the integral over $[-1.1] \setminus \tau_n(\varepsilon)$. We omit the details.

Let us mention, however, the following result in which f is replaced by $f\chi_{\Delta_n(\varepsilon)}$ with $\Delta_n(\varepsilon)$ is defined as in (2.5).

Theorem 4.4. *Let $d\alpha$ be an admissible GJ measure, $u, w \in GJ$. Assume U and V are defined as in Theorem 4.1 and satisfy* (3.8) *and* (3.9)*. Let $1 < p < +\infty$. Then*

$$\|S_n(d\alpha, f\chi_{\Delta_n(\varepsilon)})w\|_{d\alpha,p} \le c\|fu\|_{d\alpha,p} \tag{4.7}$$

for every f such that $\|fu\|_{d\alpha,p} < +\infty$ if (4.2) *and* (4.3) *hold.*

This result will be used in the next section to prove Marcinkiewicz-Zygmund type inequality. Its proof follows from the remarks above since the additional $\chi_{\Delta_n(\varepsilon)}$ allows us to use the fact that the inequality (4.6) holds for $y \in \Delta_n(\varepsilon)$, so that the proof of Theorem 4.1 can be followed through without the additional assumption on α' being nondecreasing.

5. Marcinkiewicz-Zygmund inequality

Recall the inequality for the quadrature sums in Lemma 2.7. The Marcinkiewicz-Zygmund inequality is the converse inequality (cf. [12, Theorem 2.1].

Theorem 5.1. *Let $d\alpha$ be an admissible GJ measure, β be a GJ measure such that $\beta' \in GJ2$, and u be a GJ weight function such that $u^{1-q}\alpha' \in GJ2 \cap GJ4$. Define U and V by $U^q = u^{1-q}(\alpha'\varphi)^{-q/2}\alpha'$ and $V^{-p} = \varphi^p (\alpha'\varphi)^{-p/2} \beta'$ and assume that they satisfy* (3.8) *and* (3.9) *with p and q exchanged. Let $P \in \Pi_{n-1}$ and $1 < p < +\infty$. Then*

$$\|P\|_{d\beta,p} \le c \left(\sum_{k=1}^{n} |P(x_{kn}(d\alpha))|^p u(x_{kn}(d\alpha)) \lambda_{kn}(d\alpha) \right)^{1/p} \tag{5.1}$$

provided

$$u\alpha' \ge c\beta', \qquad (\alpha'\varphi)^{-p/2} \beta' \in L^1. \tag{5.2}$$

Proof. We will write x_{kn} for $x_{kn}(d\alpha)$ in the following. Applying Lemma 2.6 and the usual duality argument gives

$$\|P\|_{d\beta,p} \le c\|P\chi_{\Delta_n(\varepsilon)}\|_{d\beta,p} = c \sup_{\|g\|_{d\beta,q}=1} \int_{-1}^{1} P(t)\chi_{\Delta_n(\varepsilon)}(t)g(t)\,d\beta.$$

By the orthogonality, the Gauss-Jacobi quadrature and the Hölder inequality, and Lemma 3.1, we have

$$\begin{aligned}
&\int_{-1}^{1} P(t)\chi_{\Delta_n(\varepsilon)}g(t)\,d\beta = \int_{-1}^{1} P(t)S_n(d\alpha;\chi_{\Delta_n(\varepsilon)}g\beta'\alpha'^{-1},t)\,d\alpha \\
&= \sum_{k=1}^{n} P(x_{kn})S_n(d\alpha;\chi_{\Delta_n(\varepsilon)}g\beta'\alpha'^{-1},x_{kn})\lambda_{kn}(d\alpha) \\
&\le c\left(\sum_{k=n}^{n} |P(x_{kn})|^p\lambda_{kn}(d\alpha)u(n,x_{kn})\right)^{1/p} \|S_n(d\alpha;g\chi_{\Delta_n(\varepsilon)}\beta'\alpha'^{-1})u^{-1/p}\|_{d\alpha,q},
\end{aligned}$$

where in the last step Lemma 2.7 is used with q and $u^{1-q}\alpha'$ in place of p and v (recall that $u^{1-q}\alpha' \in GJ2\cap GJ4$). We now apply Theorem 4.4 with $g\beta'\alpha'^{-1}$, $u^{-1/p}$, $(\beta'^{-1}\alpha')^{1/p}$, and q in place of f, w, u and p, and conclude that

$$\|S_n(d\alpha;g\chi_{\Delta_n(\varepsilon)}\beta'\alpha'^{-1})u^{-1/p}\|_{d\alpha,q} \le c\|g\|_{d\beta,q}.$$

The conditions (4.2) and (4.3) become conditions (5.2) under this substitution. This completes the proof. □

Remark 6.1. The condition $U^q \in L^1$ is not included in the condition (5.2), since it is a consequence of U^q and V^{-p} satisfying (3.8) and (3.9) with p and q exchanged. Also, $u^{1-q}\alpha' \in L^1$ is not included, since it is a consequence of $u^{1-q}\alpha' \in GJ2$.

To use this result it is necessary to choose a weight function u. We need to choose it so that $u^{1-q}\alpha' \in L^1$ (implied by $u^{1-q}\alpha' \in GJ2$), $U^q \in L^1$ and $u\alpha' \ge c\beta'$. One choice is as follows: For $d\alpha, d\beta$ being GJ measure, define

$$\sigma' = \min\{\alpha', \beta', \varphi^{-1}\} \tag{5.3}$$

and choose $u = \alpha'\sigma'^{-1}$. By the definition of GJ weight functions in 2.1, it is easy to see that $d\sigma$ is also a GJ measure. This substitution is made in the following corollary.

Corollary 5.2. *Let $d\alpha$ be an admissible $GJ\log$ measure and β' be a $GJ\log$ measure. Let $P \in \Pi_{n-1}$ and $1 < p < +\infty$. Then*

$$\|P\|_{d\beta,p} \le c\left(\sum_{k=1}^{n} |P\left(x_{kn}(d\alpha)\right)|^p\lambda_{kn}(d\sigma)\right)^{1/p}, \tag{5.4}$$

if $\Gamma_i\left((\alpha'\varphi)^{-p/2}\beta'\right) > -1$ for $1 \le i \le r$ and $(\alpha'\varphi)^{-p/2}\beta' \in L^1$.

Proof. By the definition of σ', it is evident that $\beta'\sigma'^{-1} \le c$, $\alpha'\sigma'^{-1} \le c$ and $\varphi^{-1}\sigma'^{-1} \le c$. Consequently, setting $u = \alpha'\sigma^{-1}$, the condition $u\alpha' \ge c\beta'$ holds trivially. Since $\alpha' \in GJ2$ implies that $\Gamma_i(\alpha') > -1$, the definition of σ' shows that $\Gamma_i(\sigma') > -1$. Moreover, since $u^{1-q}\alpha' \le c\sigma'$, this shows that $\Gamma(u^{1-q}\alpha') > -1$ so that $u^{1-q}\alpha' \in GJ2$. Let U be defined as in the previous theorem. Then $U^q \le c\sigma'$ and it follows that $\Gamma_i(U^q) > -1$. Furthermore, $(\alpha'\varphi)^{-p/2}\beta' \in L^1$ means that

$\varphi^{-p}V^{-p} \in L^1$, so that $-p\Gamma_i(V) > -1$ for $i = 0$ and $i = r+1$. Together with the assumption this shows that $-p\Gamma_i(V) > -1$ for all i. Consequently, using Proposition 3.5 with p and q exchanged finishes the proof. □

Our next step is to extend the Marcinkiewicz-Zygmund type inequality to include derivatives of P in the right hand side. For this we need the definition of the Hermite interpolation polynomials.

For a $m-1$ times differentiable function f, the Hermite interpolating polynomials corresponding to the distribution $d\alpha$, denoted by $H_{mn}(d\alpha, f)$, are defined to be the unique polynomial of degree at most $mn-1$ satisfying

$$H_{mn}^{(j)}(d\alpha; f, x_{kn}) = f^{(j)}(x_{kn}), \qquad 0 \le j \le m-1\ ,\ 1 \le k \le n, \tag{5.5}$$

where $x_{kn} = x_{kn}(d\alpha)$ are zeros of $p_n(d\alpha)$. When $m = 1$, $H_{mn}(d\alpha; f)$ are the Lagrange interpolating polynomials, we write $L_n(d\alpha; f) = H_{1,n}(d\alpha; f)$.

Let $d\alpha$ be an admissible GJ measure. Associated with $d\alpha$, we let v a GJ weight function such that

$$v^{-1}(x) \le c \quad \text{and} \quad v^*(x) := \alpha'(x)\varphi(x)v^{-1}(x) \le c. \tag{5.6}$$

Following the proof in [13, Theorem 3.3] we prove:

Theorem 5.3. *Let $m \ge 1$, $P \in \Pi_{mn-1}$, and $1 < p < +\infty$. Let $d\alpha$ be an admissible GJ measure, $d\beta$ be a GJ measure such that $\beta' \in GJ2$. Let v be a GJ weight function satisfying* (5.6)*. Let $u \in GJ$ such that $u^{1-q}v^{(m-1)q/2}\alpha' \in GJ2 \cap GJ4$ and $u\alpha' \in GJ2 \cap GJ4$. For $j = 1, 2, \ldots, m$, define U and V by $U^q = u^{1-q}v^{(j-1)q/2}(\alpha'\varphi)^{-q/2}\alpha'$ and $V^{-p} = \varphi^p(\alpha'\varphi)^{-jp/2}\beta'$ and assume that they satisfy* (3.8) *and* (3.9) *with p and q exchanged. Then*

$$\|P\|_{d\beta,p} \le c\left(\sum_{j=0}^{m-1}\frac{1}{n^{jp}}\sum_{k=1}^{n}\left|(\varphi(x_{kn}))^j P^{(j)}(x_{kn})\right|^p u(x_{kn})\lambda_{kn}(d\alpha)\right)^{1/p} \tag{5.7}$$

where $x_{kn} = x_{kn}(d\alpha)$, provided

$$u^{1-q}v^{(m-1)q/2}\left(\alpha'\varphi\right)^{-q/2}\alpha' \in L^1, \qquad u\alpha' \ge c\beta'(v^*)^{-(m-1)p/2}, \tag{5.8}$$

and

$$\left(\alpha'\varphi\right)^{-mp/2}\beta' \in L^1. \tag{5.9}$$

Proof. We use induction. The case $m = 1$ is precisely Theorem 5.1. Let us write $(5.8)_m$ and $(5.9)_m$ to denote the dependency of these conditions on m. We first show that $(5.8)_m$ and $(5.9)_m$ imply $(5.8)_{m-1}$ and $(5.9)_{m-1}$. For (5.8), this follows as an immediate consequence of (5.6). For (5.9), we use the fact that, for $1 \le i \le r$, if $\alpha_i'(x) \ge c$ then $(\alpha'\varphi)^{-(m-1)p/2}\beta' \le c\beta'$ on $[\bar{t}_{i-1}, \bar{t}_i]$, and if $\alpha'(x) \ge c$ then $(\alpha'\varphi)^{-(m-1)p/2}\beta' \le (\alpha'\varphi)^{-mp/2}\beta'$ on $[\bar{t}_{i-1}, \bar{t}_i]$, and similar inequalities for $i = 0$ and $r+1$.

Suppose the theorem has been proved for polynomials in $\Pi_{(m-1)n-1}$ with $m \ge 2$ and assume that $P \in \Pi_{mn-1}$. By the interpolation property of the Hermite interpolation

$$P(x) - H_{n,m-1}(d\alpha; P, x) = p_n^{m-1}(d\alpha; x)Q_n(x)$$

where $Q_n \in \Pi_{n-1}$. Using the bound of $p_n(d\alpha; x)$ on $\Delta_n(\varepsilon)$ and Lemma 2.6,

$$\begin{aligned}\|P - H_{n,m-1}(d\alpha; P)\|^p_{d\beta,p} &\le c\|(\alpha'\varphi)^{-(m-1)p/2} Q_n\|^p_{d\beta,p} \\ &\le c\sum_{k=1}^{n} |Q_n(x_{kn})|^p \big(v_n(x_{kn})\big)^{-(m-1)p/2} u(x_{kn})\lambda_{kn}(d\alpha),\end{aligned}$$

where the last inequality follows from Theorem 5.1 with $(\alpha'\varphi)^{-(m-1)p/2}\beta'$ in place of β' and $uv^{-(m-1)p/2}$ in place of u. The definition of Q shows that

$$Q_n(x_{kn}) = \frac{P^{(m-1)}(x_{kn}) - H^{(m-1)}_{n,m-1}(d\alpha; P, x_{kn})}{(m-1)!\,[p_n'(d\alpha; x_{kn})]^{m-1}}.$$

Thus, using Lemma 2.5, we can estimate the sum in two terms. The first one is bounded by, upon using $v^*(x) \le c$,

$$\frac{1}{n^{(m-1)p}} \sum_{k=1}^{n} |\left(\varphi(x_{kn})\right)^{m-1} P^{(m-1)}(x_{kn})|^p u(x_{kn})\lambda_{kn}(d\alpha),$$

which give the $j = m-1$ term in the right hand side of (5.7). The second one is bounded by,

$$\begin{aligned}\frac{1}{n^{(m-1)p}} \sum_{k=1}^{n} &|\left(\varphi(x_{kn})\right)^{m-1} H^{(m-1)}_{n,m-1}(d\alpha; P, x_{kn})|^p \\ &\times (v^*(n, x_{kn}))^{(m-1)p/2} u(x_{kn})\lambda_{kn}(d\alpha),\end{aligned}$$

which, by the inequality in Lemma 2.7 with $u\varphi^{m-1}(v^*)^{(m-1)p/2}\alpha'$ in place of v, which is in $GJ4 \cap GJ2$ since $u\alpha'$ is, is bounded by

$$\frac{c}{n^{(m-1)p}} \|\varphi^{m-1} H^{(m-1)}_{n,m-1}(d\alpha; P) u^{1/p}\|^p_{d\alpha,p} \le c\|H_{n,m-1}(d\alpha; P)u^{1/p}\|^p_{d\alpha,p},$$

where the second inequality follows from the Bernstein-Markov inequality in Lemma 2.8. Since $H_{n,m-1}(d\alpha; P) \in \Pi_{(m-1)n-1}$, by induction with $(v^*)^{(m-1)p/2}u\alpha'$ in place of β', this term is bounded by the right hand side of (5.7) with $m-1$ replaced by $m-2$. The conditions $(5.8)_{m-1}$ and $(5.9)_{m-1}$ under this substitution are implied by $u\alpha' \in L^1$, $v^*(x) \le c$ and $(5.8)_m$. Thus, we have proved that $\|P - H_{n,m-1}(d\alpha; P)\|_{d\beta,p}$ is bounded by the right hand side of (5.7). Triangle inequality

$$\|P\|_{d\beta,p} \le \|P - H_{n,m-1}(d\alpha; P)\|_{d\beta,p} + \|H_{n,m-1}(d\alpha; P)\|_{d\beta,p}$$

and induction completes the proof. $\square$

The result in the theorem is given in its general form. We can choose v and u so that the conditions become easier to check. To start with, we choose v as

$$v(x) = \max\{c, \alpha'(x)\phi(x)\}, \quad c \text{ is a constant}$$

($c = \|h\|_\infty$), which clearly satisfies (5.6). If $\alpha' \in GJ$ then $v \in GJ$. Next we define

$$\sigma' = \max\{\alpha' v^{(m-1)/2}, \varphi v^{(m-1)/2}, (v^*)^{-(m-1)/2}\beta'\}.$$

Then $\sigma' \in GJ$. We require that $d\sigma$ is a GJ measure, that is, $\sigma' \in L^1$. Since $v^*(x) \le c$, $\beta' \in L^1$ shows that this requirement put restriction on α'. For example,

if $\alpha' \in GJ\log$, then $\sigma' \in L^1$ if

$$\text{(5.10)} \quad \Gamma(\alpha') > \frac{-2}{m+1}, \quad 1 \le i \le r, \quad \text{and} \quad \Gamma(\alpha') > -\frac{1}{2} - \frac{1}{m+1}, \quad i = 0, r+1.$$

Note that the above restriction become $\Gamma(\alpha') > -1$ if $m = 1$. Furthermore, under the substitution $u = \alpha'^{-1}\sigma'$, the conditions (5.8) hold trivially. Indeed, the definition shows that $v^{(m-1)/2}\varphi^{-1}\sigma'^{-1} \le c$, $\alpha' v^{(m-1)/2}\sigma'^{-1} \le c$ and $\beta'(v^*)^{-(m-1)p/2} \le c\sigma'$. The last one is precisely the second condition in (5.8) with $u = \alpha'^{-1}\sigma'$. Moreover

$$\begin{aligned} &u^{1-q} v^{(m-1)q/2} \left(\alpha'\varphi\right)^{-q/2} \alpha' \\ &\qquad = \left(\alpha' v^{(m-1)/2}\sigma'^{-1}\right)^{q/2} \left(v^{(m-1)/2}\varphi^{-1}\sigma'^{-1}\right)^{q/2} \sigma' \le c\sigma', \end{aligned}$$

which shows that the first condition of (5.8) holds. Furthermore, we also have

$$u^{1-q} v^{(m-1)q/2} \alpha' = \left(\alpha' v^{(m-1)/2}\sigma'^{-1}\right)^{q} \sigma' \le c\sigma',$$

which shows that $u^{1-q}v^{(m-1)q/2}\alpha' \in L^1$. Finally, the substitution $u = \alpha'^{-1}\sigma'$ leads to the inequality

$$\text{(5.11)} \qquad \|P\|_{d\beta,p} \le c \left(\sum_{j=0}^{m-1} \frac{1}{n^{jp}} \sum_{k=1}^{n} \left| \left(\varphi(x_{kn})\right)^j P^{(j)}(x_{kn}) \right|^p \lambda_{kn}(d\sigma) \right)^{1/p}$$

where $x_{kn} = x_{kn}(d\alpha)$.

Hence, with these choices of u and v we can simplify the conditions in the previous theorem. This shows that (5.11) holds essentially under the condition $(\alpha'\varphi)^{-mp/2}\beta' \in L^1$ or a slightly stronger one. To make the conditions precise will require stating assumptions on U and V precisely, which can be rather involved. Instead of trying to state a general result, we restrict again to the $GJ\log$ case.

Theorem 5.4. *Let $m \ge 1$, $P \in \Pi_{mn-1}$, and $1 < p < +\infty$. Let $d\alpha$ be an admissible $GJ\log$ measure, $d\beta$ be a $GJ\log$ measure such that $\Gamma_i(\beta') > -1$, $1 \le i \le r$. Let σ' be defined as above and assume* (5.10) *so that $\sigma' \in L^1$. Then the inequality* (5.11) *holds provided $\Gamma_i\left(\alpha'^{-mp/2}\beta'\right) > -1$ for $1 \le i \le r$ and $(\alpha'\varphi)^{-mp/2}\beta' \in L^1$.*

Proof. We take $u = \alpha'^{-1}\sigma'$ in Theorem 5.3 as indicated above. Then the condition (5.8) is already satisfied. The assumption (5.10) and $\Gamma_i(\beta) > -1$ shows that $\Gamma(\sigma') > -1$ for $0 \le i \le r+1$. This implies, in particular, that $u\alpha' = \sigma' \in GJ2 \cap GJ4$ and $u^{1-q}v^{(m-1)q/2}\alpha' \in GJ2 \cap GJ4$ since the latter is bounded by $c\sigma'$. For $j = 1, 2, \ldots, m$, let U and V be defined as in Theorem 5.3 with $u = \alpha'^{-1}\sigma'$. Since $v_i(x) = c$ on $[\bar{t}_{i-1}, \bar{t}_i]$ if $\alpha_i'\varphi \le c$ and $v(x) = \alpha_i'(x)\varphi(x)$ on $[\bar{t}_{i-1}, \bar{t}_i]$ otherwise, it follows that $v^{(j-1)q/2}(x) \le cv^{(m-1)q/2}(x)$. Hence, it follows that

$$U^q(x) \le u^{1-q}v^{(j-1)q/2}(\alpha'\varphi)^{-q/2}\alpha' \le u^{1-q}v^{(m-1)q/2}(\alpha'\varphi)^{-q/2}\alpha' \le c\sigma'$$

as before, which shows that $\Gamma_i(U^p) > -1$ since $\Gamma(\sigma') > -1$. Furthermore, if $\alpha_i'\varphi \ge c$ on $[\bar{t}_{i-1}, \bar{t}_i]$ then $V^{-p} = \varphi^p(\alpha'\varphi)^{-jp/2}\beta' \le c\varphi^p\beta'$ on $[\bar{t}_{i-1}, \bar{t}_i]$, which shows that $\Gamma_i(V^{-p}) > -1$ by the assumption on $\Gamma_i(\beta')$; if $\alpha_i'\varphi \le c$ on $[\bar{t}_{i-1}, \bar{t}_i]$ then $V^{-p} \le \varphi^p(\alpha'\varphi)^{-mp/2}\beta'$ on $[\bar{t}_{i-1}, \bar{t}_i]$, which shows that $\Gamma_i(V^{-p}) > -1$ by the assumption on $\Gamma_i\left((\alpha'\varphi)^{-mp/2}\beta'\right) > -1$. By Proposition 3.5, this shows that U^q and V^{-p} satisfy (3.8) and (3.9) with p and q exchanged. $\square$

We note that the condition $\Gamma_i\left(\alpha'^{-mp/2}\beta'\right) > -1$ for $1 \le i \le r$ is just a slightly stronger than that of $(\alpha'\varphi)^{-mp/2}\beta' \in L^1$ inside $(-1,1)$, which implies $\Gamma_i\left(\alpha'^{-mp/2}\beta'\right) \ge -1$. If $\gamma_i = 0$, then the two conditions are equivalent, and the result was proved in [13].

6. Mean convergence of interpolating polynomials

With the Marcinkiewicz type inequality established, the mean convergence of the corresponding interpolating polynomials follows right away. In the following we state the result for interpolating polynomials based on the zeros of orthogonal polynomials with respect to a $GJ\log$ weight function. Let $s \ge 0$, $C^s[-1,1] = C^s$ denote the space of s times continuously differentiable functions. We begin with the following fundamental result:

Theorem 6.1. *Let $d\alpha$ be an admissible $GJ\log$ measure, $d\beta$ be a $GJ\log$ measure such that $\Gamma_i(\beta') > -1$, $1 \le i \le r$ and (5.10) holds. Assume for $0 \le \ell \le m$ that*

$$\Gamma_i\left(\alpha'^{-mp/2}\beta'\right) > -1 \quad \text{for } 1 \le i \le r \quad \text{and} \quad (\alpha'\varphi)^{-mp/2}\varphi^{\ell p}\beta' \in L^1.$$

Then for $f \in C^{m-1}$

$$\|H_{nm}(d\alpha; f)\|_{d\beta,p} \le cn^{\ell} \sum_{j=0}^{m-1} \max_{1\le k\le n} |\varphi(x_{kn})^j f^{(j)}(x_{kn})|/n^j.$$

Proof. First let $1 < p < +\infty$. Since for every fixed $d > 0$, $n^{-1} \le \varphi(x)$ on $[-1 + dn^{-2}, 1 - dn^{-2}]$, it follows from Lemma 2.6 and Lemma 2.8 that

$$\begin{aligned} \|H_{nm}(d\alpha; f)\|_{d\beta,p} &\le c\|H_{nm}(d\alpha; f)\chi_{\Delta_n(\varepsilon)}\|_{d\beta,p} \\ &\le cn^{\ell}\|H_{nm}(d\alpha; f)\varphi^t\|_{d\beta,p}. \end{aligned}$$

We then apply Theorem 5.4 with $P = H_{nm}(d\alpha; f)$ and $\varphi^{\ell p}\beta'$ in place of β'. Since the assumption implies that $\sigma' \in L^1$, it follows that

$$\sum_{k=1}^{n} \lambda_n(d\sigma; x_{kn}) \le c\int_{-1}^{1} d\sigma < +\infty.$$

This establishes the stated inequality for $1 < p < +\infty$. The case for $0 < p \le 1$ follows from an argument in [13, p. 88] which goes back to [6, p. 886]. $\square$

Evidently, one could state such a result based on Theorem 5.1 with $u = \sigma'\alpha'^{-1}$ for more general weight functions. The conditions on U and V make it less practical.

In the case $\ell = 0$, the above theorem shows the boundedness of the operator $H_{nm}(d\alpha; f)$ from $L^p(d\beta)$ to C^{m-1}. Using the Bernstein-Markov inequality, one gets also the boundedness of $\|H_{nm}(d\alpha; f)\|_{d\beta,p}$. For example, we have the following result.

Theorem 6.2. *Let $d\alpha$ be an admissible $GJ\log$ measure, $d\beta$ be a $GJ\log$ measure such that $\phi^{-kp}\beta' \in L^1$, $\Gamma_i(\beta') > -1$, $1 \le i \le r$ and (5.10) holds. Assume that*

$$\Gamma_i\left(\alpha'^{-mp/2}\beta'\right) > -1 \quad \text{for } 1 \le i \le r \quad \text{and} \quad (\alpha'\varphi)^{-mp/2}\varphi^{(m-k-1)p}\beta' \in L^1,$$

where $0 \le k \le m-1$. Then

$$\lim_{n\to\infty} \|H_{nm}^{(k)}(d\alpha; f) - f^{(k)}\|_{d\beta,p} = 0, \qquad \forall f \in C^{m-1}.$$

Proof. Since $H_{mn}(d\alpha; f)$ is a projector from C^{m-1} to Π_{mn-1}, we only need to estimate $\|H_{nm}^{(j)}(d\alpha; f - R_n)\|_{u,p}$, where R_n is a polynomial of degree n such that for $0 \le j \le m-1$,

$$\|f^{(j)} - R_n^{(j)}\|_\infty \le cE_n(f^{(m-1)})/n^{m-1-j}, \qquad \forall\ f \in C^{m-1},$$

in which $E_n(f) = \inf_{P\in\Pi_n} \|f - P\|_\infty$. Using the Bernstein-Markov inequality shows that

$$\|H_{nm}^{(k)}(d\alpha; f - R_n)\|_{d\beta,p} \le cn^k \|H_{nm}(d\alpha; f - R_n)\varphi^{-k}\|_{d\beta,p},$$

which is bounded by $cE_n(f^{(m-1)}) \to 0$ upon applying the previous theorem with $\ell = m - 1 - k$ and $\beta'\varphi^{-jp}$ in place of β'. □

In particular, for $m = 1$, this shows the convergence of Lagrange interpolation.

Corollary 6.3. *Let $d\alpha$ be an admissible $GJ\log$ measure, $d\beta$ be a $GJ\log$ measure such that $\Gamma_i(\beta') > -1$, $1 \le i \le r$. Assume that*

$$\Gamma_i\left(\alpha'^{-p/2}\beta'\right) > -1 \quad \textit{for } 1 \le i \le r \quad \textit{and} \quad (\alpha'\varphi)^{-p/2}\beta' \in L^1.$$

Then

$$\lim_{n\to\infty} \|L_n(d\alpha; f) - f\|_{d\beta,p} = 0, \qquad \forall f \in C.$$

The method also allows us to prove result concerning the best convergence order of the interpolating polynomial.

Theorem 6.4. *Let $d\alpha$ be an admissible $GJ\log$ measure, $d\beta$ be a $GJ\log$ measure such that $\phi^{-kp}\beta' \in L^1$, $\Gamma_i(\beta') > -1$, $1 \le i \le r$ and (5.10) holds. Assume that*

$$\Gamma_i\left(\alpha'^{-mp/2}\beta'\right) > -1 \quad \textit{for } 1 \le i \le r \quad \textit{and} \quad (\alpha'\varphi)^{-mp/2}\varphi^{-kp}\beta' \in L^1,$$

where $0 \le k \le m-1$. Then

$$\lim_{n\to\infty} \|H_{nm}^{(k)}(d\alpha; f) - f^{(k)}\|_{d\beta,p} \le cE_n(f^{(m-1)})/n^{m-k-1}, \qquad \forall f \in C^{m-1}.$$

These results include many special cases considered by various authors. See, for example, discussions in [5, 6, 13].

One can also apply the approach to other type of interpolation processes, for example, to Hermite-Feér interpolation polynomials and to truncated Hermite interpolation polynomials. See, for example, the discussion in [13, Section 4].

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